SS 2016

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Differential Geometry

Homework 12

Mandatory Exercise 1. (10 points) Consider a Riemannian manifold M of constant curvature K. Let $\gamma\colon [0,l] \to M$ be a geodesic parametrized by the arc length, and let $w(t)$ be any parallel vector field along γ, of length 1, and orthogonal to γ. Show that

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t) & \text{if } K > 0 \\ t w(t) & \text{if } K = 0 \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t) & \text{if } K < 0 \end{cases}$$

is a Jacobi field along γ with $J(0) = 0$ and $\nabla_{\dot\gamma} J(0) = w(0)$.

Mandatory Exercise 2. (10 points)
Let M be a Riemannian manifold with non-positive sectional curvature. Prove that for any $p \in M$ the set of points conjugate to p is empty. What does it tell you about geodesic representatives of any given homotopy class?
Hint: Assume the existence of a non-trivial Jacobi field along the geodesic $\gamma\colon [0,l] \to M$, with $\gamma(0) = p$, $J(0) = J(l) = 0$. Use the Jacobi equation to show that $\frac{d}{dt}\langle \frac{DJ}{dt}, J\rangle \geq 0$. Conclude that $\langle \frac{DJ}{dt}, J\rangle = 0$ for all t. Since $\frac{d}{dt}\langle J, J\rangle = 2\langle \frac{DJ}{dt}, J\rangle$, we have $|J|^2 = \text{const} = 0$, a contradiction.

Suggested Exercise 1. (0 points)
Let M be a Riemannian manifold, $\gamma\colon [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curves $t \mapsto f(t,s)$ are geodesics, such that $J(t) = \frac{\partial f}{\partial s}(t,0)$.
Hint: Choose a curve $\lambda(s)$, $s \in (-\varepsilon, \varepsilon)$ in M such that $\lambda(0) = \gamma(0)$ and $\dot\lambda(0) = J(0)$. Along λ choose a vector field $W(s)$ with $W(0) = \dot\gamma(0)$, $\frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$. Define $f(s,t) = \exp_{\lambda(s)} tW(s)$ and verify that $\frac{\partial f}{\partial s}(0,0) = \frac{d\lambda}{ds}(0) = J(0)$ and

$$\frac{D}{dt}\frac{\partial f}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) = \frac{DW}{ds}(0) = \frac{DJ}{dt}(0).$$

Suggested Exercise 2. (0 points)
Let $\gamma\colon [0,l] \to M$ be a geodesic and X be a Killing field on M.
(a). Show that the restriction $X(\gamma(t))$ of X to γ is a Jacobi field along γ.
(b). Show that if M is connected and there exists $p \in M$ such that $X(p) = 0$ and $\nabla_Y X(p) = 0$ for all $Y(p) \in T_pM$, then $X = 0$ on M.

Suggested Exercise 3. (0 points)
Let $K \geq 0$ be a non-negative real number and let $\rho = 1 + (\frac{K}{4}) \sum_{i=1}^n (x^i)^2$. Show that, for the Riemannian metric defined on $\mathbb{R}^n$ by

$$g_{ij}(p) = \frac{1}{\rho^2}\delta_{ij},$$

the sectional curvature is constant equal to K.

Suggested Exercise 4. (0 points)
Let M and N be Riemannian manifolds and let $f\colon M \to N$ be a diffeomorphism. Assume that N is complete and that there exists a constant $c > 0$ such that

$$|v| \geq c|df_p(v)|$$

for all $p \in M$ and all $v \in T_pM$. Prove that M is complete.

Suggested Exercise 5. (0 points)
A Riemannian manifold is said to be homogeneous if given p, $q \in M$ there exists an isometry of M which takes p to q. Prove that any homogeneous manifold is complete.

Hand in: Monday 11th July
in the exercise session
in Seminar room 2, MI